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ON TORSION OF AN ELASTIC CYLINDRICAL COSSERAT SURFACE

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Abstract—Using the linearized theory of an elastic Cosserat surface for an isotropic material, the problem of torsion of a cylindrical Cosserat surface is discussed; the edge curve perpendicular to the generator is not necessarily circular. Torque–twist relationships are obtained for both open and closed surfaces.

1. INTRODUCTION

THIS paper is concerned with the problem of torsion of an elastic Cosserat cylindrical surface; the cylindrical surface is not necessarily closed and the edge curve perpendicular to the generator need not be circular. The solution for an isotropic material is obtained with the use of an exact linearized theory contained in the paper of Green *et al.* [1].†

In the context of the linearized theory of an elastic, isotropic Cosserat surface, identification of two of the constitutive coefficients was made by Green and Naghdi [2], who compared the solution for pure bending of a flat plate with the corresponding exact solution in the three dimensional classical elasticity. Also, identification of two of the coefficients was discussed by Naghdi [3] from comparison of the solution for torsion of a circular cylindrical isotropic Cosserat surface with the corresponding exact solution in the Saint-Venant theory of torsion. For details regarding the nature of the constitutive equations of the linear theory for an isotropic elastic material, the reader is referred to [2, 3]. Additional references on the subject may be found in [4] where the relevance of the linear elastic Cosserat surface to shell theory is discussed.

In Section 2, after some preliminaries, certain results are obtained for a portion of an isotropic cylindrical Cosserat surface subjected to a resultant torque on the boundaries perpendicular to its generators. These results are then specialized to open and closed cylindrical surfaces in Sections 3 and 4, respectively. In this connection, it should be noted that E. Reissner has considered the torsion of elastic cylindrical shells (as three-dimensional bodies) in [5] and has obtained an approximate torque–twist relationship using some classical linear shell equations in conjunction with the Trefftz variational principle.

The main results here are applicable to homogeneous surfaces, but we frequently observe that they are also valid for circumferentially nonhomogeneous‡ surfaces. The solution given here is exact in the context of the linearized theory of a Cosserat surface and includes the expression for "warping" of the circumferential edge curve in the axial direction as well as the axial component of the director displacement.

† The developments in [1] are valid for a general nonlinear theory and are not limited to elastic surfaces.

‡ All references to inhomogeneity in this paper are understood to refer only to circumferential direction.

2. BASIC EQUATIONS AND PRELIMINARIES

Here we first collect the basic equations for the linear theory of an elastic cylindrical Cosserat surface. Consider a portion of a cylindrical surface and choose surface coordinates s, x along the circumferential edge boundary and along the generator. Let $\mathbf{e}_s$ and $\mathbf{e}_x$ be the base vectors along the s and x coordinate curves, respectively, and let $\mathbf{e}_3$ denote the unit normal to the surface. As in [3], we assume that the initial director $\mathbf{D}$ is coincident with the unit normal $\mathbf{e}_3$.

We denote the physical components of the infinitesimal displacement vector by $\{u_s, u_x, u_3\}$ and the physical components of the director displacement by $\{\bar{\delta}_s, \bar{\delta}_x, \bar{\delta}_3\}$ and also define the components $\{\delta_s, \delta_x, \delta_3\}$ by

$$\delta_s = \bar{\delta}_s + \left(\frac{\partial u_3}{\partial s} - \frac{u_s}{R}\right), \qquad \delta_3 = \bar{\delta}_3,$$
$$\delta_x = \bar{\delta}_x + \frac{\partial u_3}{\partial x}. \tag{2.1}$$

Since we are concerned only with cylindrical Cosserat surfaces, it is convenient to recall the basic equations from [3], where various equations for a cylindrical surface are expressed in terms of the appropriate physical components.† The surface kinematical variables are

$$e_{ss} = \frac{\partial u_s}{\partial s} + \frac{u_3}{R}, \qquad e_{xx} = \frac{\partial u_x}{\partial x},$$
$$e_{sx} = e_{xs} = \frac{1}{2}\left(\frac{\partial u_s}{\partial x} + \frac{\partial u_x}{\partial s}\right), \tag{2.2}$$

and the director kinematical variables are

$$\kappa_{ss} = \frac{\partial \bar{\delta}_s}{\partial s} + \frac{\delta_3}{R} + \frac{e_{ss}}{R}, \qquad \kappa_{xx} = \frac{\partial \bar{\delta}_x}{\partial x}, \tag{2.3}$$

$$\kappa_{xs} = \frac{\partial \bar{\delta}_x}{\partial s} + \frac{1}{R}\frac{\partial u_s}{\partial x}, \qquad \kappa_{sx} = \frac{\partial \bar{\delta}_s}{\partial x}, \tag{2.4}$$

$$\kappa_{3s} = \frac{\partial \delta_3}{\partial s} + \frac{\delta_s}{R}, \qquad \kappa_{3x} = \frac{\partial \delta_3}{\partial x}, \tag{2.5}$$

where in (2.1), (2.2), (2.3) and the equations that follow, R is a function of s.

The equilibrium equations for the case of no surface loads are

$$\frac{\partial N_{ss}}{\partial s} + \frac{\partial N_{xs}}{\partial x} + \frac{N_{s3}}{R} = 0,$$
$$\frac{\partial N_{sx}}{\partial s} + \frac{\partial N_{xx}}{\partial x} = 0, \tag{2.6}$$
$$\frac{\partial N_{s3}}{\partial s} + \frac{\partial N_{x3}}{\partial x} - \frac{N_{ss}}{R} = 0,$$

† The equations in [3] are written for a circular cylindrical surface (R = const.), but the only modification we need to make is to write $\partial/\partial s$ in place of $1/R\,\partial/\partial\theta$ and to replace subscripts θ by s. Also the order of indices in $N_{\alpha j}, M_{\alpha j}(\alpha = 1,2; j = 1,2,3)$ has been reversed in order to conform with standard notation in shell theory.

$$
\begin{aligned}
&\frac{\partial M_{ss}}{\partial s}+\frac{\partial M_{xs}}{\partial x}+\frac{M_{s3}}{R} = m_s,\\
&\frac{\partial M_{sx}}{\partial s}+\frac{\partial M_{xx}}{\partial x} = m_x,\\
&\frac{\partial M_{s3}}{\partial s}+\frac{\partial M_{x3}}{\partial x}-\frac{M_{ss}}{R} = m_3,
\end{aligned}
\tag{2.7}
$$

together with the restrictions

$$
\begin{aligned}
&N_{xs}-N_{sx} = \frac{M_{sx}}{R}, \qquad N'_{ss} = N_{ss}+\frac{M_{ss}}{R},\\
&N_{s3} = m_s-\frac{M_{s3}}{R}, \qquad N_{x3} = m_x.
\end{aligned}
\tag{2.8}
$$

In addition to the equilibrium equations (2.6), (2.7) and the equations (2.8), we have

$$
\begin{aligned}
&N_\beta = N_{\alpha\beta}\nu_\alpha, \qquad N_3 = N_{\alpha 3}\nu_\alpha,\\
&M_\beta = M_{\alpha\beta}\nu_\alpha, \qquad M_3 = M_{\alpha 3}\nu_\alpha,
\end{aligned}
\tag{2.9}
$$

where ν_α are the components of the outward unit normal vector to c bounding the surface, and N_i and M_i are the components of the force and the director force vectors acting on c (see also Fig. 1). The quantities m_i are related to $M_{\alpha j}$ through (2.7) and require constitutive

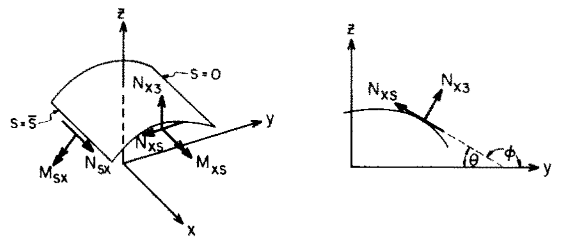

Fig. 1

equations. In (2.9), as in (2.10)–(2.13), the subscripts may be assigned the indices s and x and repeated subscripts are summed over s and x (when these are used as indices). The constitutive equations may be taken directly from [3] where they are written for an isotropic material.† Assuming isothermal deformations we have

$$N'_{\alpha\beta} = \alpha_1\delta_{\alpha\beta}e_{\lambda\lambda}+2\alpha_2 e_{\alpha\beta}+\alpha_9\delta_{\alpha\beta}\delta_3, \tag{2.10}$$

$$M_{\beta\alpha} = \alpha_5\delta_{\alpha\beta}\kappa_{\lambda\lambda}+\alpha_6\kappa_{\alpha\beta}+\alpha_7\kappa_{\beta\alpha}, \tag{2.11}$$

$$m_\alpha = \alpha_3\delta_\alpha, \qquad m_3 = \alpha_4\delta_3+\alpha_9 e_{\lambda\lambda}, \tag{2.12}$$

$$M_{\alpha 3} = \alpha_8\kappa_{3\alpha}, \tag{2.13}$$

† The constitutive equations of the linear theory of an elastic surface derived in [1] have been subjected to further simplifications in [2, 3].

where

$$N'_{xx} = N_{xx}, \qquad N'_{sx} = N_{sx}, \tag{2.14}$$

and N'_{ss} is related to N_{ss} and M_{ss} through $(2.8)_2$; also, $\delta_{\alpha\beta}$ is the Kronecker delta, and the subscripts may be assigned the indices s and x.

Guided by Saint-Venant's solution of the torsion problem, we assume that the curve sections perpendicular to the generator of the surface rotate and that they warp in the same manner for all values of x. Then u_x and δ_x are functions of s alone, as are all of $N_{\alpha j}$ and $M_{\alpha j}$. Let $\{u_y, u_z, u_x\}$ be the Cartesian components of the displacement vector and let α, a constant, denote the angle of twist per unit length. By writing $u_y = -\alpha xz$, $u_z = \alpha xy$, and by expressing the displacement vector in terms of the components $\{u_s, u_3, u_x\}$, we have (see also Fig. 1):

$$\begin{aligned} u_s &= \alpha x(y \sin \phi - z \cos \phi), \\ u_3 &= -\alpha x(y \cos \phi + z \sin \phi), \\ u_x &= \alpha w(s). \end{aligned} \tag{2.15}$$

In addition,

$$\begin{aligned} \delta_s &= \delta_3 = 0, \\ \delta_x &= \delta_x(s). \end{aligned} \tag{2.16}$$

The non-zero kinematical quantities and surface and director forces are calculated using (2.15) and (2.16) and are given by:

$$\begin{aligned} e_{sx} &= \frac{\alpha}{2}(y \sin \phi - z \cos \phi + w'), \\ \kappa_{xs} &= \delta'_x + \alpha, \qquad \kappa_{sx} = \alpha, \\ N_{sx} &= 2\alpha_2 e_{sx}, \\ M_{sx} &= \alpha_6(\delta'_x + \alpha) + \alpha\alpha_7, \\ M_{xs} &= \alpha_7(\delta'_x + \alpha) + \alpha\alpha_6, \\ N_{x3} &= m_x = \alpha_3 \delta_x, \end{aligned} \tag{2.17}$$

where $(\,)' \equiv \mathrm{d}(\,)/\mathrm{d}s$. Under these conditions the only equilibrium equations which are not identically satisfied are $(2.6)_2$ and $(2.7)_2$, and these become

$$\begin{aligned} \frac{\mathrm{d}N_{sx}}{\mathrm{d}s} &= 0, \\ \frac{\mathrm{d}M_{sx}}{\mathrm{d}s} &= N_{x3}. \end{aligned} \tag{2.18}$$

Equation $(2.18)_1$ implies that N_{sx} is a constant. Thus,

$$N_{sx} = c_1 \alpha, \tag{2.19}$$

and using $(2.17)_1$ and $(2.17)_4$ we obtain for w,

$$w(s) = c_1 \int_0^s \frac{ds}{\alpha_2} - \int_0^s (y \sin \phi - z \cos \phi)\, ds, \tag{2.20}$$

where an arbitrary additive constant has been omitted and the lower limit of integration refers to the point from which arc length is measured. Equation $(2.18)_2$ becomes, upon using $(2.17)_5$ and the last of (2.17),

$$\frac{d}{ds}[\alpha_6(\delta_x' + \alpha) + \alpha\alpha_7] = \alpha_3 \delta_x. \tag{2.21}$$

If the material is homogeneous, then $\alpha_6, \alpha_7, \alpha_3$ are constants, and (2.21) reduces to

$$\alpha_6 \delta_x'' = \alpha_3 \delta_x, \tag{2.22}$$

which has the solution†

$$\delta_x(s) = A \cosh ks + B \sinh ks, \tag{2.23}$$

where A and B are arbitrary constants and

$$k^2 = \alpha_3/\alpha_6. \tag{2.24}$$

The force resultants on the ends of the surface (i.e. at $x = \pm l$) are given by

$$\begin{aligned} F_y &= \int_0^{\bar{s}} (N_{x3} \sin \phi + N_{xs} \cos \phi)\, ds, \\ F_z &= \int_0^{\bar{s}} (-N_{x3} \cos \phi + N_{xs} \sin \phi)\, ds, \end{aligned} \tag{2.25}$$

and the resultant torque is

$$T = \int_0^{\bar{s}} [y(N_{x3} \cos \phi - N_{xs} \sin \phi) + z(N_{xs} \cos \phi + N_{x3} \sin \phi) - M_{xs}]\, ds. \tag{2.26}$$

3. TORSION OF AN OPEN SURFACE

The boundary conditions on the edges parallel to the generator of the surface are that the surface force and director force vanish. Using (2.9) with $\nu_s = 1$, $\nu_x = \nu_3 = 0$ and recalling that N_{ss}, N_{s3}, M_{ss}, M_{s3} are all zero, it is seen that we must have

$$N_{sx} = 0, \qquad M_{sx} = 0, \tag{3.1}$$

on $s = 0$, $s = \bar{s}$. Thus,

$$\begin{aligned} c_1 &= 0, \\ A &= \frac{\alpha(\alpha_6 + \alpha_7)}{k\alpha_6} \tanh\left(\frac{k\bar{s}}{2}\right), \\ B &= -\frac{\alpha}{k\alpha_6}(\alpha_6 + \alpha_7). \end{aligned} \tag{3.2}$$

† If segments $s_1 \leq s \leq s_2$ of the surface are homogeneous, then (2.23) still holds, but A, B, k will vary from section to section. But if k^2 is the same in each segment, then (2.23) holds for all s.

With the use of (2.20) and $(3.2)_1$ the warping of the surface is

$$w(s) = -\int_0^s (y \sin \phi - z \cos \phi)\, ds. \tag{3.3}$$

To evaluate the resultant forces and torque we need the geometric formulae

$$\cos \phi = \frac{dy}{ds}, \qquad \sin \phi = \frac{dz}{ds}, \qquad \frac{d\phi}{ds} = \frac{1}{R}. \tag{3.4}$$

Then the forces F_y and F_z can be evaluated by using $(2.18)_2$ and $(2.8)_1$ in (2.25), integrating by parts the first term in each expression, and using $c_1 = 0$ and $(3.4)_3$. The result, as it is to be expected, is that F_y and F_z vanish identically. Using a similar procedure on (2.26) and noting $(3.4)_{1,2}$, we obtain for the torque the expression

$$T = -\int_0^{\bar{s}} (M_{sx} + M_{xs})\, ds. \tag{3.5}$$

Now from $(2.17)_{5,6}$, (2.23) and (3.2),

$$M_{sx} + M_{xs} = \alpha(\alpha_6 + \alpha_7)\left\{2 + \frac{(\alpha_6 + \alpha_7)}{\alpha_6}\left[\tanh\left(\frac{k\bar{s}}{2}\right)\sinh ks - \cosh ks\right]\right\}, \tag{3.6}$$

and for a homogeneous surface the torque is

$$T = -2\alpha(\alpha_6 + \alpha_7)\left[\bar{s} - \frac{(\alpha_6 + \alpha_7)}{k\alpha_6}\tanh\left(\frac{k\bar{s}}{2}\right)\right]. \tag{3.7}$$

This result is exact in the context of theory used and involves no approximation.† However, if we identify the Cosserat surface with the midsurface of a thin elastic shell (regarded as a three-dimensional body) and identify α_6, α_7, k with the corresponding quantities in shell theory,‡ then for a "slit tube", the first term in brackets in (3.7), being a good approximation to the quantity in the brackets, corresponds to Reissner's result in [5].

4. TORSION OF A CLOSED SURFACE

For a closed surface, the resultant forces may be shown to vanish identically in a manner similar to that in Section 3 (except that it is no longer necessary to put $c_1 = 0$). Moreover from (2.26), the torque reduces to

$$T = -\int_0^{\bar{s}} (M_{sx} + M_{xs})\, ds - 2c_1 \alpha A_c, \tag{4.1}$$

where we have used $(3.4)_{1,2}$ and where A_c is the area bounded by the edge curve of the surface in the y–z plane.

We now restrict attention to homogeneous surfaces. Then (2.23) is the solution for δ_x. However, for a closed surface we must have a single-valued director displacement at $s = 0$, $s = \bar{s}$, where $\bar{s}$ is again the length of the perimeter of the cross-section. Since (2.23)

† Note that in (3.7), T depends on $(\alpha_6 + \alpha_7)$, α_6 as well as α_3 in view of (2.24).

‡ In making such an identification with results obtained from three-dimensional classical elasticity, we should put $\alpha_6 = \alpha_7$.

is not single-valued, $A = B = 0$ and there is no director displacement, i.e.,

$$\delta_x = 0. \tag{4.2}$$

The shear force N_{sx} is not zero for a closed surface (recall that it is the boundary condition of zero force along the edges $s = 0$, $s = \bar{s}$ which causes N_{sx} to be zero for an open surface). N_{sx} can be found by noting (2.19), where c_1 is evaluated by the requirement that $w(s)$, the "warping" function of the surface be single-valued. Calculating $w(\bar{s})$ from (2.20) by means of $(3.4)_{1,2}$ and putting $w(\bar{s}) = w(0) = 0$, we easily find that†

$$c_1 = \frac{2A_c}{\int_0^{\bar{s}} \mathrm{d}s/\alpha_2}, \tag{4.3}$$

and N_{sx} is given by (2.19).

The torque–twist relation can be found from (4.1) by first using $(2.17)_{5,6}$ and (4.2) to get

$$M_{sx} + M_{xs} = 2\alpha(\alpha_6 + \alpha_7). \tag{4.4}$$

Now (4.1) reduces to

$$T = -\left[\frac{4\alpha_2 A_c^2}{\bar{s}} + 2(\alpha_6 + \alpha_7)\bar{s}\right]\alpha. \tag{4.5}$$

This relation depends only on the constitutive coefficients α_2 and $(\alpha_6 + \alpha_7)$.

If we restrict attention to a circular cylindrical surface, the torque–twist relation is

$$T = -2\pi R^2\left[\alpha_2 + \frac{2(\alpha_6 + \alpha_7)}{R^2}\right]\alpha R, \tag{4.6}$$

where R is the radius of the circle. This expression was obtained by Naghdi [3] and was used by him to identify the coefficients α_2 and $\alpha_6 + \alpha_7$ for the case of a thin elastic shell,

$$\alpha_2 = Gh, \qquad \alpha_6 + \alpha_7 = \frac{Gh^3}{6}, \tag{4.7}$$

where h is the thickness of the shell and G is the shear modulus.

If we take the coefficients as given in (4.7), and if we take the Cosserat surface to be the mid-surface of a thin elastic shell regarded as a three-dimensional body (so that A_c is now the area enclosed by the mid-surface of the shell), then (4.5) corresponds with the result given by E. Reissner [5].

Equation (4.5) is easily applied to calculate torque–twist relations for cross-sections other than circular (e.g. an ellipse), but we do not pursue this further.

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† Equation (4.3) holds for any nonhomogeneity in the circumferential direction in the surface.

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Абстракт—Используя линеаризованную теорему упругой поверхности Коссера для изотропного материала, определяется задача кручения цилиндрической поверхности Коссера. Краевая кривая, перпендикулярна к образующей не обязательно круглая. Получаются зависимости для момента и кручениа так для открытых, как и замкнутых поверхностей.